©*The Journal of Risk and Insurance*, 1999, Vol. 66, No. 2, 219-237

# The Value of Guarantees on Pension Fund Returns

George G. Pennacchi


## Abstract

Contingent claims analysis is used to value government guarantees associated with defined contribution pension plans. Values are derived for two types of guarantees on the rate of return earned by an individual pension fund: a guarantee of a fixed minimum rate of return and a guarantee of a minimum rate of return that is set relative to the performance of other pension funds. The value of a minimum pension benefit guarantee for a participant in a mandatory defined contribution pension plan is also derived. Values for each of these guarantees are illustrated using typical parameter values. Martingale pricing theory provides a unifying framework for valuing these guarantees by either an explicit formula or numeric computation using a Monte Carlo simulation.


Social Security reform has been a serious concern to many countries. In Latin America, a number of reforms have been implemented by partially or fully privatizing pension obligations. Proposals to reform the United States Social Security system have also included privatization features.[1] Most often, these privatization reforms have encouraged or required that individuals switch from a government-run defined benefit pension plan to a privately run defined contribution system. A potential obstacle

George G. Pennacchi is Professor of Finance, Department of Finance, at the University of Illinois. A non-technical description of this research as well as a review of defined benefit pension guarantees is presented in "Government Guarantees for Old Age Income," Chapter 9 of *Prospects for Social Security Reform,* edited by Olivia S. Mitchell, Robert J. Myers, and Howard Young, University of Pennsylvania Press, 1999. The author is grateful for comments and guidance by Patrick Brockett, William Murphy, Hector Salazar, Salvador Valdés-Prieto, Krishna Ramaswamy, Nicholas Souleles, and, especially, Olivia Mitchell.

[1] For example, Feldstein and Samwick (1998) propose to augment the existing defined benefit Social Security system with Personal Retirement Accounts. Two percent of an individual's earnings would be contributed to these private, defined contribution retirement accounts. Also, a proposal by the Clinton Administration would allocate government funds for Universal Savings Accounts, which are defined contribution retirement accounts for low- and moderate-income individuals that would be separate from the current Social Security system.

exists, however, in gaining political approval for this type of reform. By converting to a defined contribution system, individuals may be exposed to risks not previously faced in a government-sponsored defined benefit plan. Participants in a defined contribution system risk experiencing lower than anticipated investment returns, possibly leaving them with inadequate wealth during their retirement years.[2]

To make reforms involving a conversion to a defined contribution system more attractive to the public, governments have typically provided guarantees that reduce individuals' exposure to investment risks. As a result, guarantees of defined contribution pensions have recently become more common, especially in Latin America, which has been at the forefront of pension privatizations.[3] These guarantees differ from the more traditional government guarantees of defined benefit, private pension funds, such as those provided by the United States' Pension Benefit Guaranty Corporation. Defined contribution guarantees have been of two main types. One type insures the periodic rates of return earned by the pension funds in which individuals can invest. Typically, this takes the form of a guarantee that each defined contribution pension fund earns an annual rate of return greater than a pre-specified minimum. The second type of guarantee directly insures each individual's, rather than each pension fund's, return on pension savings. This type of guarantee ensures that participants in a defined contribution system receive a minimum pension payment throughout their retirement years, even if their pension savings are exhausted because of withdrawals during their retirement.

Because governments usually retain an insurance obligation following a pension privatization, estimating the value of government guarantees is important for gauging the implicit subsidy associated with a particular pension reform. By accounting for the cost of guarantees in government budget statistics, an improved, market value-based measure of fiscal spending can be obtained. In addition, these cost estimates could make feasible a system of risk-based insurance premiums that would reduce or eliminate the net subsidy from providing guarantees.

Previous research on valuing pension guarantees has focused on defined benefit guarantees, with little analysis devoted to guarantees on defined contribution pension plans.[4] But with the growing popularity of defined contribution pensions and their critical role in many recent pension reforms, research analyzing defined contribution guarantees is clearly needed. This article derives a number of new results for valuing these guarantees using "contingent claims" analysis. It illustrates that the

[2] Bodie, Marcus, and Merton (1988) discuss the relative merits of defined contribution and defined benefit pension plans. Defined contribution plans are increasingly popular throughout the world. In some countries, such as Denmark and Singapore, they are the primary source of pension savings. Other countries, such as Australia, Chile, and Switzerland, require that a portion of pension savings be of the defined contribution type. For discussions of various countries' pension systems, see Mitchell (forthcoming), Davis (1996), and Turner and Wantanabe (1995).

[3] For descriptions and critical analyses of Latin American pension reforms, see Mitchell and Barreto (1997) and Queisser (1995).

[4] This research emphasis is likely a result of the historical dominance of defined benefit pensions plans in most developed countries. Research on valuing defined benefit guarantees includes Marcus (1987), Hsieh, Chen, and Ferris (1994), Pennacchi and Lewis (1994), and Lewis and Pennacchi (1999).

"martingale pricing" technique for calculating contingent claims values can be a unifying framework for valuing many kinds of guarantees. This technique may yield explicit formulas for guarantee values, or it can allow for numeric valuation by Monte Carlo simulation.

Defined contribution pension guarantees bear some similarities to guarantees made by insurance companies on the minimum maturity cash value of equity-linked life insurance policies. Research on such insurance policy guarantees includes Brennan and Schwartz (1976), Boyle and Schwartz (1977), and Banicello and Ortu (1993). However, guarantees on a pension fund's periodic rates of return differ, because they are a series of sequential guarantees, not a single guarantee based on a maturity value. Also, a guarantee on an individual's minimum pension benefit differs from these insurance policy guarantees, because its value depends on the individual's wage contributions during his or her working years. Hence, wage uncertainty is a critical source of risk affecting this pension guarantee.

When governments guarantee private contracts, such as pension plans, adverse selection and moral hazard problems may arise. These incentive problems can be alleviated by properly structuring and pricing guarantees and/or regulating the activities of the parties on whose behalf the guarantee is given. Discussions of these important issues can be found in a number of recent papers and, because of a lack of space, will not be repeated here.[5] Because this study's focus is on valuing guarantees, it often takes the equilibrium risk decisions of the participating parties as given. But it should be emphasized that these decisions are often linked to the guarantee's structure, pricing, or regulation.[6] In some cases, by estimating the costs of guarantees and then charging appropriate risk-based insurance premiums that cover these costs, adverse selection and moral hazard problems can be reduced or eliminated.

The article is organized into three sections. The first considers two types of pension rate of return guarantees: a fixed rate of return guarantee and a rate of return guarantee that is relative to the performance of other pension funds. The second section considers a guarantee of a minimum pension benefit for a participant in a mandatory defined contribution pension plan. Values for these rates of return and minimum pension guarantees are illustrated using typical parameter values. A concluding section follows.

## VALUING GUARANTEES ON A PENSION FUND'S RATES OF RETURN

This section addresses two types of pension fund rate of return guarantees made by governments. These guarantees can be valued by recognizing their similarity to various types of "exotic" options, such as "forward start options," "options to exchange one asset for another," and "options on the minimum of two risky assets."[7] This article begins by considering a rather simple fixed minimum rate of return guarantee, similar to one provided by Uruguay. It then considers a minimum rate of return guarantee that is a function of the average rate of return earned by all pension funds. The

---

5 See Bodie and Merton (1993), Pesando (1996), and Smalhout (1996).

6 This study does recognize the effect of incentives on the equilibrium value of guarantees. For example, in the case of the minimum pension guarantee, the author chooses to model a retiree's incentive-compatible choice of pension-payment options.

7 Hull (1997) describes and analyzes these options.

modeling of this second guarantee is based on one provided by the government of Chile. A similar guarantee is made by the government of Colombia and analyzed in recent independent research by Fischer (1998).[8]

### A Minimum Fixed Rate of Return Guarantee

Uruguay permits both private and public pension funds, known as "Asociaciones de Fondos de Ahorro Previsional" (AFAP).[9] In the case of public AFAPs (but not private AFAPs), the government guarantees pension fund participants a minimum annual real rate of return of 2 percent. Thus, a public AFAP that earns less than 2 percent during a given year would require a government transfer to make up the difference.

Assume that the instantaneous real rate of return on a public AFAP's securities, $dS/S$, follows the process

$$dS / S = \alpha_s dt + \sigma_s dz_s . \tag{1}$$

Here, $\alpha_s$ is the expected rate of return on the AFAP's securities, $\sigma_s$ is the standard deviation of the rate of return on its security portfolio, and $dz_s$ is a standard Wiener process. Here $\sigma_s$ is assumed to be constant but $\alpha_s$ could, in general, be changing stochastically. Further, it is assumed that there exists an asset paying a constant, default-free, real rate of return equal to $r$.[10]

Consider a guarantee of a minimum fixed real rate of return equal to $m$, such as $m = .02$. If this guarantee is made starting at the current date 0 and ending at date $\tau$, then its value is given by a standard Black-Scholes put option with an exercise price $X = Se^{m\tau}$, where $S$ denotes the current value of the AFAP's securities. Denoting the value of this guarantee as $H$,

$$\begin{aligned} H &= Xe^{-r\tau} N(-d_2) - SN(-d_1) \\ &= Se^{m\tau - r\tau} N(-d_2) - SN(-d_1) \\ &= S\left[e^{(m-r)\tau} N(-d_2) - N(-d_1)\right] \\ &\equiv Sh(\tau) \end{aligned} \tag{2}$$

where $d_1 = (r - m + \frac{1}{2}\sigma_s^2)\tau / (\sigma_s \sqrt{\tau})$ and $d_2 = d_1 - \sigma_s \sqrt{\tau}$. Because the exercise price is proportional to $S$, the current value of the AFAP's securities, the option value (guarantee) is also proportional to $S$.

[8] Fischer (1998) values Colombia's guarantees using a discrete-time binomial model. Many of his findings and those from this article's continuous-time model of Chilean guarantees are qualitatively similar, although this article originated prior to and independently from the publication of his results.

[9] See Mitchell (1996) for a discussion of pension-system reform in Uruguay.

[10] This assumption of a constant real interest rate is a simplification that allows the application of the Black-Scholes framework. If the real interest rate is assumed to change randomly, it would represent another source of uncertainty in addition to $S$. While the martingale pricing technique that we use can be generalized to handle this case, an explicit solution for the value of the guarantee cannot then be derived. However, in this situation, the guarantee value can be computed by a Monte Carlo simulation (where risk-adjusted processes for the real interest rate and $S$ are simulated), similar to that which is done in the next section.

Now consider a minimum rate of return guarantee that begins at some future date (year) y and lasts for $\tau$ periods. Hence the guarantee is for a minimum rate of return of m over the period y to $y+\tau$. Let the current date be 0 and denote the current value of this guarantee as $H(0,y,\tau)$. This type of guarantee is analogous to a "forward start" option: the exercise price of this guarantee, $X = S(y)e^{m\tau}$, is proportional to the value of the future AFAP's portfolio at the start of the guarantee period, $S(y)$. Although this exercise price is unknown as of the current date, this type of option can be valued using the "risk neutral" technique of Cox and Ross (1976), a technique that Harrison and Kreps (1979) generalized into "martingale pricing" theory. In the absence of arbitrage opportunities, it can be shown that a "risk neutral" probability measure exists such that

$$\begin{aligned} H(0,y,\tau) &= e^{-ry}\hat{E}_0[S(y)h(\tau)] \\ &= e^{-ry}h(\tau)\hat{E}_0[S(y)] \end{aligned} \tag{3}$$

where $\hat{E}_0[\cdot]$ is the date 0 expectation taken under the risk neutral measure.[11] In other words, the expectation is computed under the assumption that the rate of return on the AFAP's securities equals the risk-free rate, that is, $\alpha_s = r$. In this case $\hat{E}_0[S(y)] = Se^{ry}$, where $S$ is the current date 0 value of the pension fund securities.

More generally, if it is also supposed that the pension fund is growing because of net new contributions at a proportional real growth rate of $g$, then[12] $\hat{E}_0[S(y)] = Se^{gy+ry}$. Substituting this into equation (3), the date 0 value of a guarantee for the period $y$ to $y + \tau$ is

$$H(0,y,\tau) = h(\tau)Se^{gy}. \tag{4}$$

If a government makes this guarantee on an annual basis ($\tau$=1) for $n$ consecutive years, the total value of the guarantee, $H_n$, is

$$H_n = Sh(1)\sum_{y=0}^{n-1} e^{gy}. \tag{5}$$

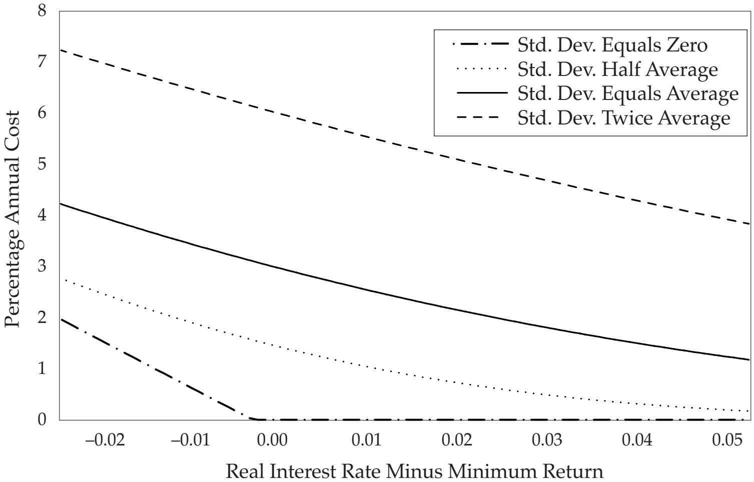
If the annual guarantee's value is strictly positive, that is, $h(1) > 0$, and the real growth rate of the fund is non-negative, $g \geq 0$, the value of this guarantee grows without bound as $n \to \infty$. An implication is that governments should be cautious when deciding whether to make such long-dated guarantees, especially if pension funds are expected to grow substantially.

Note that the value of this guarantee is a function of the difference between the risk-free rate and the guaranteed rate of return, $r - m$, as well as the rate of return standard deviation of the pension fund, $\sigma_s$. Figure 1 plots the annual percentage cost of the guarantee, $100 \times h(1)$, as a function of $r - m$ for four different values of $\sigma_s$. The first value, $\sigma_s = 0$, reflects the case in which the AFAP invests entirely in risk-free real assets, earning a certain rate of return equal to $r$. The next three cases reflect risky

11 Kocić (1997) reviews how the martingale pricing condition results from an absence of arbitrage. A more detailed and technical discussion of martingale pricing can be found in Duffie (1996).

12 Contributions are assumed to be made at the start of each guarantee period. Thus, if $S(y^-)$ is the value of the pension fund's investments just prior to a guarantee period that begins at date $y$, then the contribution equals $S(y^-)[e^{g\tau} - 1]$.

**FIGURE 1**
Value of Fixed Rate of Return Guarantee by Difference Between Real Interest Rate and Minimum Return




AFAP investments. Because social security reform was enacted in Uruguay only in 1995, and data on AFAP returns are not yet available, the non-zero values of $\sigma_s$ used here reflect a parameter estimate taken from Chilean pension fund returns, namely $\sigma_s = 0.038$, 0.077, and 0.154, which represents one half, once, and twice the average of all Chilean pension funds. This was estimated from annual data on Chilean pension fund returns for the period 1981-92, as reported in Diamond and Valdés-Prieto (1994, p. 300).

As expected, Figure 1 shows that the cost of the guarantee rises with the volatility of the AFAP's investments, $\sigma_s$, and decreases as the difference between the real interest rate and the minimum guarantee, $r - m$, widens. Note that the function becomes more convex as volatility decreases, and for the case of $\sigma_s = 0$, the relation is kinked at $r - m = 0$. This limiting case reflects the common-sense result that if the AFAP invests entirely in risk-free assets earning $r$, then the value of the guarantee equals zero for $m \leq r$, but the value of the guarantee is non-random and equal to $100 \times (m - r)$ for $m > r$.

## A Minimum Relative Rate of Return Guarantee

In Chile, private pension funds, known as "Administradora de Fondos de Pensiones" (AFPs), are each required to earn an annual real rate of return that is a function of the average annual real rate of return earned by all of Chile's private pension funds. If $R_a$ is the (*ex post*) average annual rate of return earned by all AFPs, then each AFP must earn at least $\min[R_a - \alpha, \beta R_a]$ where $\alpha = 0.02$ and $\beta = 1/2$.[13] Thus, if $R_a$ exceeds 4 percent, then each AFP must earn at least $1/2\, R_a$. For smaller values of $R_a$, each AFP must earn at least $R_a - 2$ percent. All AFPs are required to hold capital (a guarantee fund) of at least 1 percent of the value of its pension portfolio, invested in the same security

[13] Peru has a pension system similar to Chile's and requires this same minimum rate of return formula for its AFPs. A similar system is also found in Argentina, where the pension fund rate of return requirement is of this form, but with $\alpha = 0.02$ and $\beta = 0.7$ (Mitchell and Barreto, 1997).

portfolio as that of its pension fund. If the fund's return is less than min[$R_a - \alpha$, $\beta R_a$], then it must make up the difference from its capital and replenish its capital within 15 days. The AFP's license would be revoked if it fails to do so.[14] Thus, given an AFP capital ratio of $c = 0.01$, the government would be exposed to loss following an AFP that earns less than min[$R_a - \alpha$, $\beta R_a$] $- c =$ min[$R_a - \alpha - c$, $\beta R_a - c$].

To value this guarantee for a given AFP, it is assumed that the rate of return on its security portfolio is given by the process in equation (1) above and the average rate of return of all AFPs is $dS_a/S_a$. Here, $S_a$ is assumed to follow a similar process

$$dS_a \,/\, S_a = \alpha_a\, dt + \sigma_a\, dz_a \tag{6}$$

where $dz_s dz_a = \rho dt$, so that $\rho$ is the instantaneous correlation between the individual AFP's portfolio return and the return of all AFPs.[15]

Based on equation (6), consider the following two "reference" funds having values $X$ and $V$, respectively. As will be demonstrated, these reference funds provide benchmarks for determining the payoffs of guarantees.

*Reference Fund 1*:

$$dX \,/\, X = (\alpha_a - q_x)dt + \sigma_a\, dz_a \tag{7}$$

*Reference Fund 2*:

$$\begin{aligned} dV \,/\, V &= (\beta\,\alpha_a - c)dt + \beta\,\sigma_a\, dz_a \\ &= (\alpha_v - q_v)dt + \beta\,\sigma_a\, dz_a \end{aligned} \tag{8}$$

Note that the process for $X$ is equal to that of $S_a$ but with a rate of return smaller by $q_x$. In the case of Chile, $q_x$ will equal $\alpha + c = 0.03$. The process for $V$ is similar to $S_a$ except that its increments are a proportion $\beta$ times those of $S_a$, less a further decline in growth of $c$. In the absence of arbitrage, it can be shown that the drift of equation (8) can be written as $\alpha_v - q_v$ where $\alpha_v = \alpha_a - (1-\beta)\varphi\sigma_a$, $q_v = (1-\beta)(\alpha_a - \varphi\sigma_a) + c = (1-\beta)r(t) + c$, where $\varphi$ is the market price of risk from the $dz_a$ process and $r(t)$ is the risk-free interest rate.[16]

The payoff from the government's guarantee for a given AFP can now be written in terms of equations (1), (7), and (8). Note that the rate of return on $X$ is $\alpha + c = 0.03$ less

[14] As discussed in Diamond and Valdés-Prieto (1994), this capital requirement reduces the potential moral hazard arising from the government's guarantee. The incentive for an AFP's owners to take risks that differ significantly from other AFPs is curtailed, since they would be first in line to lose their capital investment should the AFP's return fall below the minimum. In addition, moral hazard is constrained by government regulations that limit an AFP's proportions of specific types of securities. In practice, there has been relatively little diversity in the investments and portfolio returns of Chilean AFPs.

[15] Note that if all individual AFPs follow a process of the form of equation (1), the average rate of return will not exactly conform to equation (6). However, the form of equation (6) will be a close approximation to the true average if the individual AFP is sufficiently small relative to the total. This is similar to the common practice of modeling a constant volatility diffusion process for both individual stocks and stock indices.

[16] If we write $\alpha_a = r + \varphi\sigma_a$, then in the absence of arbitrage the expected rate of return on $V$ must be $\alpha_v = r + \varphi\beta\sigma_a$. Substituting for $r$, $\alpha_v = \alpha_a - (1-\beta)\varphi\sigma_a$. Thus if $\beta\alpha_a - c = \alpha_v - q_v$, substituting for $\alpha_v$ gives $q_v = (1-\beta)(\alpha_a - \varphi\sigma_a) + c = (1-\beta)r + c$.

than the AFP average while that of $V$ is proportion $\beta = 1/2$ of the AFP average, less $c = 0.01$. Thus, the value of the government's relative rate of return pension guarantee starting at date $y$ and lasting for $\tau$ periods is analogous to a European option to exchange the individual AFP pension assets, $S(y+\tau)$, and obtain $\min[X(y+\tau), V(y+\tau)]$, given that at date $y$ the funds have the same value as the AFP's portfolio, that is, $S(y) = X(y) = V(y)$. This guarantee resembles an option to exchange one risky asset for (a function of) another.

Following Margrabe (1978), it is useful to normalize the reference funds by $S$. Define $x = X/S$ and $v = V/S$. Applying Itô's lemma, they satisfy:

$$\begin{aligned} dx / x &= (\alpha_a - q_x - \alpha_s + \sigma_s^2 - \sigma_a \sigma_s \rho)dt + \sigma_a dz_a - \sigma_s dz_s \\ &= (\alpha_1 - q_x)dt + \sigma_1 dz_1 \end{aligned} \tag{9}$$

where $\alpha_1 \equiv \alpha_a - \alpha_s + \sigma_s^2 - \sigma_a \sigma_s \rho$ and $\sigma_1^2 \equiv \sigma_a^2 + \sigma_s^2 - 2\rho\sigma_a\sigma_s$.

$$\begin{aligned} dv / v &= (\alpha_v - q_v - \alpha_s + \sigma_s^2 - \beta\sigma_a \sigma_s \rho)dt + \beta\sigma_a dz_a - \sigma_s dz_s \\ &= (\alpha_2 - q_v)dt + \sigma_2 dz_2 \end{aligned} \tag{10}$$

where $\alpha_2 \equiv \alpha_v - \alpha_s + \sigma_s^2 - \beta\sigma_a\sigma_s\rho$ and $\sigma_2^2 \equiv \beta^2\sigma_a^2 + \sigma_s^2 - 2\rho\beta\sigma_a\sigma_s$. This implies that $dz_1 dz_2 = \rho_{12} dt$ where $\rho_{12} = (\beta\sigma_a^2 - \sigma_a\sigma_s\rho(1+\beta) + \sigma_s^2)/(\sigma_1\sigma_2)$.

With this normalization, there exists a risk-neutral probability measure where the value of the guarantee equals $S(y)$ times a European call option written on $\min[x(y+\tau), v(y+\tau)]$ where $x$ and $v$ have the dividend yields $q_x$ and $q_v$, respectively, the risk-free interest rate is zero and the exercise price is 1. This zero interest rate and unit exercise price are the result of the normalization by $S$.

Now suppose that y is the current date, with $S$, $X$, and $v$ being the current (known) values of the three assets. Then the value of this call option on the minimum of two assets was shown by Stulz (1982) and Johnson (1987) to equal

$$\begin{aligned} S\Bigg[ & xe^{-q_x\tau} N_2\left(\gamma_1 + \sigma_1\sqrt{\tau}, \frac{\ln(v/x) + (q_x - a_v - .5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \frac{\rho_{12}\sigma_2 - \sigma_1}{\sigma}\right) \\ & + ve^{-q_v\tau} N_2\left(\gamma_2 + \sigma_2\sqrt{\tau}, \frac{\ln(x/v) + (q_v - q_x - .5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \frac{\rho_{12}\sigma_1 - \sigma_2}{\sigma}\right) \\ & - N_2(\gamma_1, \gamma_2, \rho_{12})\Bigg] \end{aligned} \tag{11}$$

where $N_2(\cdot,\cdot,\cdot)$ is the bivariate normal distribution function,

$\gamma_1 = \left[\ln(x) - (q_x + \tfrac{1}{2}\sigma_1^2)\tau\right] / \left(\sigma_1\sqrt{\tau}\right)$, $\gamma_2 = \left[\ln(v) - (q_v + \tfrac{1}{2}\sigma_2^2)\tau\right] / \left(\sigma_2\sqrt{\tau}\right)$, and where

$\sigma^2 \equiv \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_a^2(1-\beta)^2$, so that $\sigma = \sigma_a(1-\beta)$.

Since a rate of return guarantee over the period $y$ to $y+\tau$ implies that the date $y$ values of $S$, and the reference funds $X$ and $V$ will all be equal, then $x = v = 1$. With this, the above option value simplifies to

$$S\left[e^{-q_x\tau}N_2\left(\frac{-q_x}{\sigma_1\sqrt{\tau}}+\tfrac{1}{2}\sigma_1\sqrt{\tau},\frac{q_x-q_v}{\sigma\sqrt{\tau}}-\tfrac{1}{2}\sigma\sqrt{\tau},\frac{\rho_{12}\sigma_2-\sigma_1}{\sigma}\right)\right.$$
$$+\ e^{-q_v\tau}N_2\left(\frac{-q_v}{\sigma_2\sqrt{\tau}}+\tfrac{1}{2}\sigma_2\sqrt{\tau},\frac{q_v-q_x}{\sigma\sqrt{\tau}}-\tfrac{1}{2}\sigma\sqrt{\tau},\frac{\rho_{12}\sigma_1-\sigma_2}{\sigma}\right)$$
$$\left.-\ N_2\left(\frac{-q_x}{\sigma_1\sqrt{\tau}}-\tfrac{1}{2}\sigma_1\sqrt{\tau},\frac{-q_v}{\sigma_2\sqrt{\tau}}-\tfrac{1}{2}\sigma_2\sqrt{\tau},\rho_{12}\right)\right]$$
$$\equiv\ Sh(\tau). \tag{12}$$

As in the previous example of the fixed minimum rate of return guarantee, we see that this guarantee is also proportional to $S$, the current value of the AFP's securities. Thus, using the same analogy to a forward start option, the date 0 value of a guarantee for the period $y$ to $y+\tau$, denoted $H(0,y,\tau)$, equals $h(\tau)Se^{gy}$, where $g$ is the growth rate of net new contributions to the pension fund. If a government makes this guarantee on an annual basis ($\tau = 1$) for $n$ consecutive years, the total value of the guarantee, $H_n$, is

$$H_n = Sh(1)\sum_{y=0}^{n-1} e^{gy}. \tag{13}$$

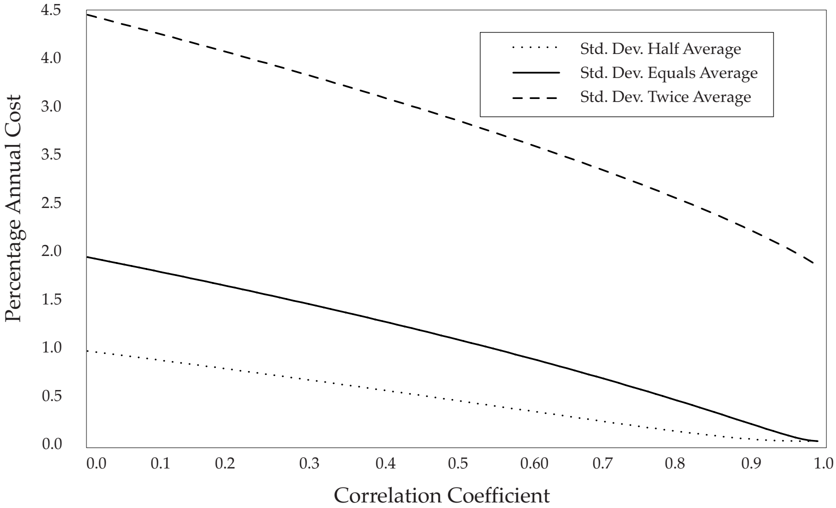
As with the case of the fixed rate of return guarantee, the value of this relative rate of return guarantee becomes unbounded as $n \to \infty$ if $h(1) > 0$ and $g \geq 0$.

Figure 2 plots the value $100 \times h(1)$, the annual percentage cost of this guarantee, for various correlations between individual AFP and average AFP returns, $\rho$. The figure assumes $\sigma_a = 0.077$, $\alpha = 0.02$, $\beta = 1/2$, $c = 0.01$, and $d = 0.04$. The guarantee value is shown for three cases, when the individual AFP return standard deviation equals ($\sigma_s = \sigma_a$), is twice ($\sigma_s = 2\sigma_a$), or is one-half ($\sigma_s = 1/2\,\sigma_a$) that of the average of AFPs. As would be expected, for each of these three cases the value of the guarantee falls as the correlation rises. Interestingly, when $\sigma_s \leq \sigma_a$, the values converge to zero as the correlation coefficient becomes 1, and the lower is $\sigma_s$ relative to $\sigma_a$, the lower is the guarantee value. However, if $\sigma_s > \sigma_a$, which should be the case for the typical AFP since individual risk is diversified by the average, then even when the correlation is perfect, the guarantee will have positive value.

## VALUING MINIMUM PENSION GUARANTEES FOR DEFINED CONTRIBUTION PLANS

This section analyzes the value of a guarantee of a minimum pension benefit for a participant in a mandatory defined contribution pension system, where a fixed proportion of the participant's wage is assumed to be contributed to a pension fund that earns risky returns. Two studies estimated the value of this guarantee for the case of Chile. Wagner (1991), whose results are summarized in Diamond and Valdés-Prieto (1994), values this guarantee by simulating its annual cost for an economy at a steady state with respect to the demographic profile of pension participants. The model calculates this cost under different assumptions regarding the real rate of return on pension fund assets and the minimum pension benefit guaranteed to retirees. Another study by Zarita (1994) applies contingent claims pricing to value Chile's minimum pension benefit. His model explicitly allows for a stochastic rate

**FIGURE 2**
Value of Relative Rate of Return Guarantee by Correlation Between Individual and Average AFP




of return on pension fund assets, so that a worker's accumulated pension savings at retirement are random. If the worker's savings at retirement are less than the cost of an annuity that would provide the minimum pension benefit, the government is assumed to make a payment to cover the difference. The risk-neutral expected value of this government payment is calculated using a Monte Carlo simulation of the worker's risky pension investment assuming a deterministic level of wage contributions each period and a constant real interest rate.

The approach used in this section is similar to that of Zarita (1994) but includes a number of extensions. First, in addition to allowing pension fund returns to be stochastic, a worker's real wage, and thus the worker's monthly pension contribution, is also allowed to follow a random process. The evolution of real wages is also assumed to influence the minimum pension set by the government when the worker retires. Second, real interest rates are assumed to follow a stochastic process. This is potentially important because random real interest rates add uncertainty to the value of the worker's retirement annuity, since annuity values depend on the contemporaneous real interest rate. Thus, interest rate uncertainty will also affect the discounted value of the government's payment, since, in general, these payments will be systematically related to not only pension investment returns and wage levels, but also the real interest rate.

Third, the government's payments are modeled to provide a retiree with a minimum pension benefit in a different, arguably more realistic, manner. Upon reaching retirement, a retiree may have a choice regarding benefit payments. With sufficient pension savings, a Chilean retiree may close his or her pension account and use the savings to purchase a lifetime annuity providing a benefit at or above the minimum pension. Alternatively, the retiree's pension account can be maintained, and benefits can be received by a scheduled withdrawal of funds from the account. For a retiree with an account balance insufficient to purchase a minimum pension annuity, a scheduled withdrawal of funds is required. The maximum amount that a retiree can withdraw each year is determined by a government formula that depends on the retiree's current pension account balance and the value of a lifetime annuity, where this annuity is calculated using the government's "technical" interest rate. If a retiree's pension account balance is exhausted, the government will begin paying him or her the minimum pension.

As discussed in Turner and Wantanabe (1995) and Smalhout (1996), a person who reaches retirement with a pension balance that is slightly above or at the price of a minimum pension annuity has an incentive to not purchase an annuity but to choose the scheduled withdrawal option. By choosing this phased withdrawal, a person receives free longevity insurance at the government's expense. Should the person live longer than expected, the government would need to provide a minimum pension. If, instead, life is shorter than expected, the person's heirs would inherit the balance of the pension account. Thus, in some states of the world, the government pays a subsidy that would not occur if a lifetime annuity were immediately purchased. Hence, for those reaching retirement with moderate to small pension savings, who are those most likely to require minimum pension assistance, it is more realistic to assume a scheduled withdrawal of pension funds. The model of this section, unlike Zarita (1994), explicitly models this scheduled withdrawal.

Following is a brief description of the model. More details of and justification for the model can be found in the Appendix. The model assumes three main (continuous-time) stochastic processes: the rate of return on pension fund assets, the growth in real wages, and the change in the short-term, real interest rate. These three processes may, in general, be correlated. This short term real interest rate determines the term structure of real yields based on the Vasicek (1977) model. An additional minor source of uncertainty is the individual's mortality. The probability of death at each age is assumed to be uncorrelated with economic variables and is taken from Chile's official life table. A hypothetical male worker is assumed to begin making pension contributions at age 20 and, should he live until the retirement age of 65, begin a scheduled withdrawal of his pension savings at the maximum level allowed by law. The worker's monthly contribution equals 10 percent of his randomly evolving wage, which is invested in his pension fund earning a random rate of return.

At retirement, the maximum that can be withdrawn each month is calculated following the actual Chilean government formula, which is described in Diamond and Valdés-Prieto (1994, p. 290):

> Every twelve months, the fixed real amount that will be withdrawn in each of the following twelve months is calculated. This amount is $P = F/UC$, where $F$ is the current balance in the individual account and $UC$ is calculated from the official life table and a technical interest rate ($TR$), and it is essentially the reserve needed to finance an annuity that pays \$1 a month when investments yield $TR$. The return $TR$ in turn is calculated according to a formula fixed by law. This formula specifies that for AFP $i$, $TR_i$ for year $t = 0.2 \times$(average of past real returns of Fund $i$ during past five years) + $0.8 \times$(average of implicit rates of return on all real annuities sold in calendar year $t - 1$).

The model here follows this formula exactly, except that in calculating $TR$, "the implicit rate of return on all real annuities sold in calendar year $t - 1$" is approximated with the date $t$ real yield on a nine-year zero-coupon indexed bond, since Diamond and Valdés-Prieto (1994, p. 296) report that the duration of newly issued annuities is approximately nine years. Thus, during the individual's retirement period, the amount withdrawn is a function of the last five years' returns of the individual's AFP (affecting $TR$), the current randomly evolving real yield on a nine-year bond (affecting $TR$), the individual's age (affecting $UC$), and the individual's pension fund balance (which is affected by past withdrawals and pension asset returns).

The above formula's "maximum" withdrawal is, however, truly the maximum only if it exceeds the government's minimum pension level. If not, the amount withdrawn is equal to the minimum pension. This occurs until the retiree's pension account is exhausted, should he or she live that long. After the account balance is exhausted, the government pays the minimum pension until the end of the retiree's life.

For simplicity, the model assumes that the minimum pension set by the government at the beginning of the individual's retirement is determined by the following formula: minimum pension = $1/4 \times$ (average wage at start of individual's working life) $\times$ (growth in real wages over the individual's working life) $\times 1/2$. This formula reflects the likelihood that the government would tend to raise the minimum pension should real wages (and the standard of living) rise. Since Turner and Wantanabe (1995, p. 210) report that the minimum pension is approximately 25 percent of the average wage and because the model here assumes that the individual's real wage will almost double over his or her 45 years of work (1.5 percent average annual growth), the formula should maintain an approximate 25 percent minimum pension-average wage ratio.[17]

The stochastic processes for pension assets, $S$, real wages, $W$, and the real interest rate, $r$, are of the form

$$dS\ /\ S = \alpha_s\, dt + \sigma_s\, dz_s \tag{14}$$

$$dW\ /\ W = (\alpha_w - c_w)dt + \sigma_w\, dz_w \tag{15}$$

$$dr = \beta(\gamma - r)dt + \sigma_r\, dz_r \tag{16}$$

where the $dz_i$'s, $i = s,w,r$, are three, possibly correlated, Wiener processes. Let $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$, $i,j \varepsilon \{s,w,r\}$ be the instantaneous covariances between the three processes in equations (14) through (16). Here, $\alpha_s$ is the expected rate of return and $\sigma_s$ is the standard deviation of the rate of return on the security portfolio. Also, $\sigma_w$ is the (assumed constant) standard deviation of the percentage change in the wage rate while $\alpha_w$ is the expected rate of return that would be earned on a (hypothetical) security that has the same rate of change risk ($\sigma_w dz_w$) as the wage rate. Since wages, $W$, will not necessarily be expected to grow at the rate of an asset that bears its same risk, $c_w$ denotes

[17] One component of an individual's real wage growth would reflect average (economy-wide) real wage increases, and another component would reflect real wage increases because of greater productivity or seniority during the individual's career. Thus, individual real wage growth might be expected to exceed the economy-wide average. In any case, the simulations result in an average minimum pension at the individual's retirement date equal to 44.7 percent of the initial average real wage.

the difference between this equilibrium expected asset return and the expected growth in wages.[18]

Regarding the interest rate process in equation (16), $\beta$ is a measure of how quickly $r(t)$ is expected to return to its long-run mean of $\gamma$ following a deviation. Here, $\sigma_r$ is the standard deviation of the unexpected changes in the interest rate. Vasicek (1977) showed that if $r(t)$ follows equation (16), then the current (date $t$) price of a bond that matures in $\tau$ periods is

$$P(t,\tau) = A(\tau)e^{-B(\tau)r} \tag{17}$$

where $B(\tau) \equiv \dfrac{1-e^{-\beta\tau}}{\beta}$, $A(\tau) \equiv \exp\left[(B(\tau)-\tau)\left(\gamma + q\dfrac{\sigma_r}{\beta} - \tfrac{1}{2}\dfrac{\sigma r^2}{\beta^2}\right) - \dfrac{\sigma_r^2 B(\tau)^2}{4\beta}\right]$,

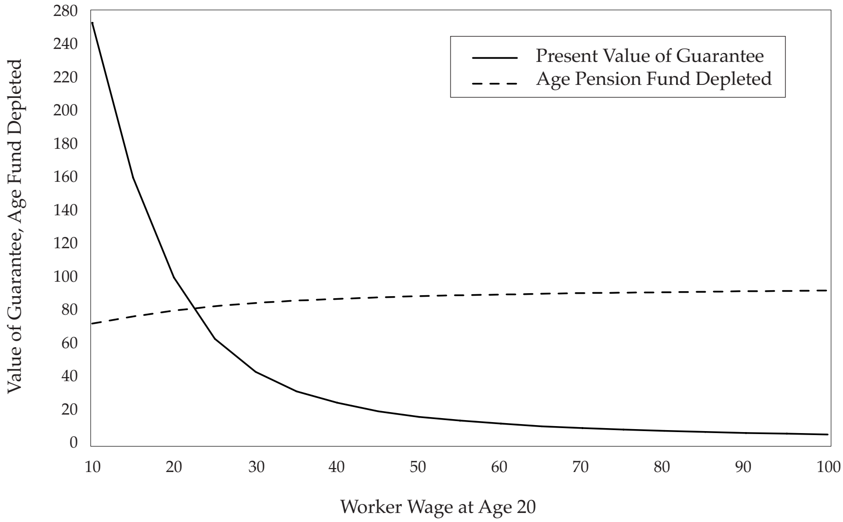
and $q$ is the equilibrium market price of interest rate risk. This implies that the expected rate of return on this $\tau$ period bond is equal to $r(t) + q\sigma_r B(\tau)$, where $\sigma_r B(\tau)$ is the bond's rate of return standard deviation and $q\sigma_r B(\tau)$ is its risk premium.

The Appendix provides more details regarding the present value calculation of the minimum pension guarantee. In particular, it shows how the three stochastic processes, equations (14) through (16), can be transformed to their risk-adjusted counterparts so that the value of the guarantee can be calculated as an expectation of the discounted government payments needed to cover the minimum pension. It also derives the discrete-time means and covariances of these continuous-time processes so that one can calculate the expectation of government payments using a Monte Carlo simulation, where contributions or withdrawals from the individual's pension fund account occur each month. The Monte Carlo technique for computing guarantees extends the work of Boyle (1977) and Cooperstein, Pennacchi, and Redburn (1995).

To illustrate how values of this minimum pension guarantee can be calculated, specific parameter values were selected. These parameter values may not be realistic. The article's goal is to illustrate the qualitative features of this guarantee rather than provide the most accurate estimates. So $\sigma_s$ was set equal to 0.077, which, as mentioned earlier in section I, was estimated from Chilean pension fund returns over the period 1981-92. Based, in part, from estimates in Foresi, Penati, and Pennacchi (1997), the real interest rate parameters were set at $\beta = 0.1$, $\gamma = 0.035$, $\sigma_r = 0.015$, and $q = 0.09$. For the real wage process, an annual standard deviation of $\sigma_w = 0.01$ was assumed, and $c_w$ was set to 0.02. The effect of the $c_w$ calibration is that if the risk of wages changes

[18] The parameter $c_w$, which affects the minimum pension guarantee's value, can be statistically identified if one assumes that asset returns satisfy an equilibrium asset pricing model. For example, in the standard, single-factor capital asset pricing model (CAPM), $\alpha_w = r + \beta_w(\alpha_m - r)$, where $\alpha_m$ is the expected real rate of return on the market portfolio of all securities and $\beta_w = \sigma_{wm}/\sigma_m^2$ is the covariance between the growth in real wages and the real rate of return on the market portfolio divided by the variance of the real rate of return on the market portfolio. Estimates of $\beta_w$ and $\alpha_m$ can be obtained from historical data using standard methods, with $\beta_w$ estimated from regressing real wage growth in excess of the real risk-free rate on the real excess rate of return on the market portfolio. From the above CAPM relation, an estimate of $\alpha_w$ is determined. An estimate of $c_w$ will then equal this estimate of $\alpha_w$ minus the expected growth in real wages. The expected growth in real wages, $(\alpha_w - c_w)$, can be estimated as the historical average real wage growth.

**Figure 3**
Minimum Pension Guarantee by Initial Wage Level




is non-priced risk, so that the equilibrium value of $\alpha_w = r(t)$, then the average growth in real wages would be $\gamma - c_w = 0.035 - 0.02 = 0.015$. If a risk premium is associated with wage changes, this average growth rate may be different. Finally, the correlation structure of $\rho_{sw} = \rho_{wr} = 0.2$, and $\rho_{wr} = -0.2$ was assumed.

Guarantee values were calculated for the case of a 20-year-old male, beginning wage earner starting with a zero pension fund balance. Mortality was based on the Chilean life tables for male annuitants. Assuming an average Chilean monthly real wage of 100 at the time this individual begins work, the average level of the minimum pension set by the government (according to the formula discussed above) at the worker's retirement date was 44.7.

Figure 3 graphs the present values of the minimum pension guarantee for this 20-year-old worker for different initial monthly wages ranging between 10 and 100.[19] The value of this guarantee ranges from 251.8 for an individual with a monthly wage of 10 to 5.8 for an initial monthly wage of 100. The shape of the relationship is convex as one might expect given the put option-like nature of this guarantee. Also plotted in Figure 3 is the individual's age at which his pension fund account would be depleted, given that he lives that long. This age ranges from 72.1 for an initial wage of 10 to 91.8 for an initial wage of 100. Note that this age profile has a concave shape. Higher initial wages increase the time before the pension account is depleted, but less than proportionally. While higher initial wages tend to result in proportionally higher accumulated pension savings at retirement, the government's scheduled withdrawal formula allows greater pension withdrawals for individuals with higher savings. Thus, the withdrawal schedule tends to subdue the effect that greater retirement savings have on the age at which pension funds are depleted.

[19] A GAUSS program that calculates the guarantee values in Figure 3 is available from the author upon request.

## CONCLUSIONS

Many actual and proposed social security reforms seek to privatize pension obligations by requiring that individuals contribute to defined contribution pension plans. However, when contributions to these pension plans are mandatory, individuals are subject to investment risks that they previously did not face in a government-sponsored defined benefit plan. To make privatization reforms politically attractive to the public, governments typically offer guarantees that reduce individuals' exposure to investment risks.

Recent advances in contingent claims analysis provide important insights for valuing pension guarantees. This article illustrates how the martingale pricing approach can be applied to value a variety of guarantees on pension fund returns. Perhaps the most attractive feature of this approach is the relatively few assumptions needed to calculate guarantee values. The main restriction imposed by this approach is that equilibrium asset prices do not allow for arbitrage opportunities.

This article analyzes guarantees at a microeconomic level. It considers the values of defined contribution rate of return guarantees for individual pension funds and the value of a minimum pension guarantee for an individual worker in a defined contribution pension system. A system of risk-based insurance premiums could be based on these individual guarantee values, so as to reduce the government's subsidy (and the subsidy's economic distortions) from providing guarantees. To calculate a government's total liability from its guarantees, one can then aggregate over individual pension funds or types of workers.[20] This aggregate value could be incorporated into measures of government spending, thereby providing a more accurate indicator of fiscal policy.

## APPENDIX

### Valuing Contingent Claims by Monte Carlo Simulation of Equivalent Martingale Probability Measures

Let $F(t)$ be the date $t$ value of a contingent claim, in particular, a guarantee of a minimum pension for a worker making mandatory wage contributions to a defined contribution pension fund. The future payoff on this contingent claim is assumed to depend on multiple state variables, specifically, the values of a portfolio of securities and a worker's wage, denoted by $S(t)$ and $W(t)$, respectively. In addition, it is assumed that interest rates are stochastic. Denote the interest rate on a very short (instantaneous) maturity default free bond as $r(t)$. The three state variables, $S(t)$, $W(t)$, and $r(t)$, are assumed to follow the processes given in equations (14) to (16) of the text. Given the interest rate process in equation (16), Vasicek (1977) showed that equilibrium price of a bond that matures in $\tau$ periods will then be given by equation (17).

[20] A similar aggregation is performed in Cooperstein, Pennacchi, and Redburn (1995), where the aggregate value of deposit insurance is calculated by aggregating the values of deposit insurance provided to individual banks.

The absence of arbitrage implies the existence of an equivalent martingale measure such that the value of the contingent claim, $F(t,S(t),W(t),r(t))$, equals

$$F(t) = \tilde{E}_t \left[\exp\left[-\int_t^T r(\mu)d\mu\right]F\left(T,\ S(T),W(T),R(T)\right)\right] \tag{A.1}$$

where $\tilde{E}_t\ [\cdot]$ is the date $t$ expectation of the discounted contingent claim at date $T$ where this expectation is taken under the "risk-neutral" martingale probability measure. What this means is that the expectation of the date $T$ distribution of $\exp[-\int_t^T r(\mu)d\mu]F(T)$ is that which is generated by the processes in equations (14) through (16) but with the following two transformations. First, the expected rates of return on all assets equal the risk free rate, that is, $\alpha_s = \alpha_w = r(t)$. Second, the process followed by the risk-free rate $r(t)$ should be that in (16) but where $\gamma$ is replaced with

$\Gamma \equiv \gamma + q\dfrac{\sigma_r}{\beta}$. A proof that $F(t)$ can be valued by taking expectations of

$\exp[-\int_t^T r(\mu)d\mu]F(T)$ is given in Duffie (1996).

Computing the expectation of $\exp[-\int_t^T r(\mu)d\mu]F(T)$ can be carried out as follows. Define $s(t) \equiv \ln[S(t)]$ and $w(t) = \ln[W(t)]$, and let $x(t) = [s(t)\ w(t)\ r(t)]'$ be the $3 \times 1$ vector of these state variables. Over any finite period of time, $x(t)$ will be distributed as a trivariate normal under the risk-neutral probability measure. Specifically, over a time period of length $\tau$, the continuous-time process for $x(t)$ is equivalent to the trivariate normal discrete-time AR(1) process:

$$x(t+\tau) = \theta(\tau) + \varphi(\tau)x(t) + \varepsilon \tag{A.2}$$

where the $3 \times 1$ vector $\theta(\tau) \equiv \begin{bmatrix} \left(\Gamma - \tfrac{1}{2}\sigma_s^2\right)\tau - \Gamma B(\tau) \\ \left[\Gamma - \left(c_w + \tfrac{1}{2}\sigma_w^2\right)\right]\tau - \Gamma B(\tau) \\ \Gamma\beta B(\tau) \end{bmatrix}$ and the $3 \times 3$ matrix

$\varphi(\tau) \equiv \begin{bmatrix} 1 & 0 & B(\tau) \\ 0 & 1 & B(\tau) \\ 0 & 0 & e^{-\beta\tau} \end{bmatrix}$. Thus, $\theta(\tau) + \varphi(\tau)x(t)$ gives the discrete-time means of the three

state variables. Here $\varepsilon$ is a $3 \times 1$ vector of mean-zero normally distributed random variables with a $3 \times 3$ covariance matrix equal to $Q \equiv \int_0^\tau \varphi(\mu)\ \Sigma\varphi(\mu)'\ d\mu$, where $\Sigma$ is the continuous-time processes' instantaneous covariance matrix whose $i,j^{th}$ element is given by $\sigma_{ij}$. Carrying out this integration, the discrete-time variances of the three state variables are

$$Q_{ss} = \sigma_s^2\tau + 2\sigma_{sr}\left[\tau - B(\tau)\right]/\beta + \sigma_r^2\left[\tau - B(\tau) - \tfrac{1}{2}\ \beta B(\tau)^2\right]\ /\beta^2 \tag{A.3}$$

$$Q_{ww} = \sigma_w^2\tau\ + 2\sigma_{wr}\left[\tau - B(\tau)\right]/\beta + \sigma_r^2\ \left[\tau - B(\tau) - \tfrac{1}{2}\beta B(\tau)^2\right]/\beta^2 \tag{A.4}$$

$$Q_{rr} = \tfrac{1}{2}\sigma_r^2\ \left(1 - e^{-2\beta\tau}\right) \tag{A.5}$$

and their covariances equal

$$Q_{sw} = \sigma_{sw}\tau + (\sigma_{sr} + \sigma_{wr})\,[\tau - B(\tau)]\;/\;\beta + \sigma_r^2[\tau - B(\tau) - \tfrac{1}{2}\beta B(\tau)^2]\;/\;\beta^2 \qquad \text{(A.6)}$$

$$Q_{sr} = \sigma_{sr} B(\tau) + \tfrac{1}{2}\sigma_r^2 B(\tau)^2 \qquad \text{(A.7)}$$

$$Q_{wr} = \sigma_{wr} B(\tau) + \tfrac{1}{2}\sigma_r^2\; B(\tau)^2. \qquad \text{(A.8)}$$

The expectation in (A.1), $\hat{E}_t\left[\exp\left[-\int_t^T r(\mu)d\mu\right]\; F(T, x(T))\right]$, can now be computed by taking the sample average of the discounted date $T$ payoffs resulting from a large number of outcomes of the process (A.2) simulated using a random number generator. For many problems, it is necessary to model periodic cashflows into or out of asset portfolios when simulating these payoffs. This can be done as long as these cashflows are a function of current or past state variables. For example, if a worker makes a mandatory contribution of 10 percent of wages to a pension fund at the end of each month, the beginning of month value of $x$(t) over a period $\tau = 1/12$ year would be simulated using (A.2), and then 0.10 ·exp[$w(t + \tau)$] would be added to exp[$s(t + \tau)$] before simulating $x(t + \tau)$ over the next month. In a similar manner, cash outflows, such as when a retired worker makes pension fund withdrawals, can be modeled as long as these withdrawals are a function of the current or past state variables. This Monte Carlo simulation technique involving cashflows which occur at discrete points in time is developed in Cooperstein, Pennacchi, and Redburn (1995).

Moreover, the date at which the contingent claims pays off, $T$, can also be a function of the state variables, for example, the date when a retired worker's withdrawals have depleted his or her pension fund balance. Payoffs will then be discounted by the realized sequence of short-term interest rates, $\exp[-\int_t^T r(\mu)d\mu]$. For example, if there were $n$ sub-periods of length $\tau$ between date $t$ and date $T$, $\int_t^T r(\mu)d\mu$ can be computed as $\tau\sum_{i=0}^{n\tau} r(t+i) - \tfrac{1}{2}\tau[r(t) + r(T)]$. In other words, the integral is computed using the average short rate over each subinterval between dates $t$ and $T$.

## REFERENCES


Banicello, Anna Rita, and Fulvio Ortu, 1993. "Pricing Equity-Linked Life Insurance with Endogenous Minimum Guarantees," *Insurance: Mathematics and Economics* 12, 245-257.

Bodie, Zvi, and Robert C. Merton. 1993. "Pension Benefit Guarantees in the United States: A Functional Analysis." In Ray Schmidt, ed., *The Future of Pensions in the United States*. University of Pennsylvania Press.

Bodie, Zvi, Alan J. Marcus, and Robert C. Merton. 1988. "Defined Benefit versus Defined Contribution Pension Plans: What are the Real Trade-offs?" In Zvi Bodie, John B. Shoven, and David A. Wise, eds., *Pensions in the U.S. Economy*. University of Chicago Press.

Boyle, Phelim. 1977. "Options Pricing: A Monte Carlo Approach." *Journal of Financial Economics* 4, 323-38.

Boyle, Phelim., and Eduardo S. Schwartz. 1977. "Equilibrium Prices of Guarantees Under Equity-Linked Contracts." *Journal of Risk and Insurance* 44, 639-80.

Brennan, Michael J., and Eduardo S. Schwartz. 1976. "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee." *Journal of Financial Economics* 3, 195-213.

Cooperstein, Richard L., George G. Pennacchi, and F. Stevens Redburn. 1995. "The Aggregate Cost of Deposit Insurance: A Multiperiod Analysis." *Journal of Financial Intermediation* 4, 242-71.

Cox, John, and Stephen Ross. 1976. "The Valuation of Options for Alternative Stochastic Processes." *Journal of Financial Economics* 3, 145-166.

Davis, E. Philip. 1996. "An International Comparison of the Financing of Occupational Pensions." In Zvi Bodie, Olivia S. Mitchell, and John A. Turner, eds., *Securing Employer Based Pensions: An International Perspective*. Pension Research Council and University of Pennsylvania Press.

Diamond, Peter, and Salvador Valdés-Prieto. 1994. "Social Security Reforms." In Barry P. Boxworth, Rudiger Dornbusch, and Raul Laban, eds., *The Chilean Economy: Policy Lessons and Challenges*. The Brookings Institution, Washington, D.C.

Duffie, Darrell. 1996. *Dynamic Asset Pricing Theory*. Princeton University Press.

Feldstein, Martin, and Andrew Samwick. 1998. "Two Percent Personal Retirement Accounts: Their Potential Effects on Social Security Tax Rates and National Saving." National Bureau of Economic Research Working Paper No 6540.

Fischer, Klaus P. 1998. "A Discrete Martingale Model of Pension Fund Guarantees in Colombia: Pricing and Market Effects." CRÉFA Working Paper No 98-02, Université Laval.

Foresi, Silverio, Alessandro Penati, and George Pennacchi. 1997. "Indexed Bonds and the Inflation Risk Premium." Working Paper, University of Illinois.

Harrison, Michael, and David Kreps. 1979. "Martingales and Arbitrage in Multi-period Securities Markets." *Journal of Economic Theory* 20: 381-408.

Hsieh, Su-Jane, Andrew H. Chen, and Kenneth R. Ferris. 1994. "The Valuation of PBGC Insurance Using An Option Pricing Model." *Journal of Financial and Quantitative Analysis* 29, 89-99.

Hull, John. 1997. *Options, Futures, and Other Derivative Securities*, Third Edition. Prentice Hall, Englewood Cliffs, NJ.

Johnson, Herbert. 1987. "Options on the Maximum and Minimum of Several Assets." *Journal of Financial and Quantitative Analysis* 22, 277-83.

Kocić, Aleksandar. 1997. "Numeraire Invariance and Generalized Risk Neutral Valuation." *Advances in Futures and Options Research* 9, 157-73.

Lewis, Christopher M., and George G. Pennacchi. 1998. "Valuing Insurance for Defined-Benefit Pension Plans." *Advances in Futures and Options Research* 10 (forthcoming).

Marcus, Alan. 1987. "Corporate Pension Policy and the Value of PBGC Insurance." Chapter 3 in *Issues in Pension Economics*, Z. Bodie, J. Shoven, and D. Wise, eds. University of Chicago Press.

Margrabe, William. 1978. "The Value of an Option to Exchange One Asset For Another." *Journal of Finance* 33, 177-86.

Mitchell, Olivia S. forthcoming. "Retirement Savings in the Developed and Developing World: Institutional Structure, Economics Effects, and Lessons for Economies in Transition." In Avril Van Adams, Elizabeth King, and Zafiris Tzannatos, eds., *Labor Market Policies for Managing the Social Cost of Economic Adjustment*. World Bank, Washington, D.C.

Mitchell, Olivia S. 1996. "Social Security Reform in Uruguay: An Economic Assessment." PRC WP 96-20, The Wharton School, University of Pennsylvania.

Mitchell, Olivia S., and Flavio Ataliba Barreto. 1997. "After Chile, What? Second-Round Social Security Reforms in Latin America." *Revista de Anâlisis Econômico* 12, 3-36.

Pennacchi, George G., and Christopher M. Lewis. 1994. "The Value of Pension Benefit Guaranty Corporation Insurance." *Journal of Money, Credit and Banking* 26, 735-53.

Pesando, James E. 1996. "The Government's Role in Insuring Pensions." In Zvi Bodie, Olivia S. Mitchell, and John A. Turner, eds., *Securing Employer Based Pensions: An International Perspective*. Pension Research Council and University of Pennsylvania Press.

Queisser, Monika. 1995. "Chile and Beyond: The Second-Generation Pension Reforms in Latin America." *International Social Science Review* 48, 23-39.

Smalhout, James H. 1996. *The Uncertain Retirement*. Irwin Publishing. Chicago.

Stulz, Rene. 1982. "Options on the Minimum or Maximum of Two Assets." *Journal of Financial Economics* 10, 161-85.

Turner, John A., and Noriyasu Wantanabe. 1995. *Private Pension Policies in Industrialized Countries: A Comparative Analysis*. W. E. Upjohn Institute for Employment Research. Kalamazoo, Michigan.

Vasicek, Oldrich A. 1977. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics* 5, 177-88.

Wagner, Gert. 1991. "La Sequridad Social y el Programa de Pensión Mínima Garatizada." *Estudios de Economía* 18, 35-91.

Zarita, Salvador. 1994. "Minimum Pension Insurance in the Chilean Pension System." *Revista de Anâlisis Econômico* 9, 105-26.